THE CARMICHAEL NUMBERS UP TO 10^{17}

RICHARD G.E. PINCH

ABSTRACT. We extend our previous computations to show that there are 585355 Carmichael numbers up to 10^{17}. As before, the numbers were generated by a back-tracking search for possible prime factorisations together with a "large prime variation". We present further statistics on the distribution of Carmichael numbers.

1. INTRODUCTION

A *Carmichael number* N is a composite number N with the property that for every b prime to N we have $b^{N-1} \equiv 1 \mod N$. It follows that a Carmichael number N must be square-free, with at least three prime factors, and that $p-1|N-1$ for every prime p dividing N: conversely, any such N must be a Carmichael number.

For background on Carmichael numbers and details of previous computations we refer to our previous paper [4]: in that paper we described the computation of the Carmichael numbers up to 10^{15} and presented some statistics. These computations have since been extended to 10^{16} [5] and now to 10^{17}, using similar techniques, and we present further statistics.

The complete list of Carmichael numbers up to 10^{17} is available from the author.

2. ORGANISATION OF THE SEARCH

We used improved versions of strategies described in [4].

The principal search was a depth-first back-tracking search over possible sequences of primes factors $p_1, \ldots, p_d$. Put $P_r = \prod_{i=1}^{r} p_i$, $Q_r = \prod_{i=r+1}^{d} p_i$ and $L_r = \operatorname{lcm}\{p_i - 1 : i = 1, \ldots, r\}$. We find that Q_r must satisfy the congruence $N = P_r Q_r \equiv 1 \bmod L_r$ and so in particular $Q_d = p_d$ must satisfy a congruence modulo L_{d-1}: further $p_d - 1$ must be a factor of $P_{d-1} - 1$. We modified this to terminate the search early at some level r if the modulus L_r is large enough to limit the possible values of Q_r, which may then be factorised directly.

We also employed the variant based on proposition 2 of [4] which determines the finitely many possible pairs (p_{d-1}, p_d) from P_{d-2}. In practice this was useful only when $d = 3$ allowing us to determine the complete list of Carmichael numbers with three prime factors up to 10^{18}. The count of these is 35586 (not 35585 as incorrectly reported in our preliminary work).

Finally we employed a different search over large values of p_d, in the range $10^7 < p_d < 10^9$, using the property that $P_{d-1} \equiv 1 \bmod (p_d - 1)$ and factorising the numbers in this arithmetic progression less than $10^{18}/p_d$.

3. STATISTICS

We have shown that there are 585355 Carmichael numbers up to 10^{17}, all with at most 11 prime factors. We let $C(X)$ denote the number of Carmichael numbers less than X and $C(d, X)$ denote the number with exactly d prime factors. Table 1 gives the values of $C(X)$ and Table 2 the values of $C(d, X)$ for X in powers of 10 up to 10^{17}.

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n	$C(10^n)$
3	1
4	7
5	16
6	43
7	105
8	255
9	646
10	1547
11	3605
12	8241
13	19279
14	44706
15	105212
16	246683
17	585355

TABLE 1. Distribution of Carmichael numbers up to 10^{17}.

X	3	4	5	6	7	8	9	10	11	total
3	1	0	0	0	0	0	0	0	0	1
4	7	0	0	0	0	0	0	0	0	7
5	12	4	0	0	0	0	0	0	0	16
6	23	19	1	0	0	0	0	0	0	43
7	47	55	3	0	0	0	0	0	0	105
8	84	144	27	0	0	0	0	0	0	255
9	172	314	146	14	0	0	0	0	0	646
10	335	619	492	99	2	0	0	0	0	1547
11	590	1179	1336	459	41	0	0	0	0	3605
12	1000	2102	3156	1714	262	7	0	0	0	8241
13	1858	3639	7082	5270	1340	89	1	0	0	19279
14	3284	6042	14938	14401	5359	655	27	0	0	44706
15	6083	9938	29282	36907	19210	3622	170	0	0	105212
16	10816	16202	55012	86696	60150	16348	1436	23	0	246683
17	19539	25758	100707	194306	172234	63635	8835	340	1	585355

TABLE 2. Values of $C(X)$ and $C(d,X)$ for $d \leq 10$ and X in powers of 10 up to 10^{17}.

We have used the same methods to calculate the smallest Carmichael number S_d with d prime factors for d up to 34. The results are given in Table 3. Heuristics suggest that a good approximation for $\log S_d$ might be $2 \log 2(d-1) \log(d-1)$. There is some support for this from Table 4.

In Table 5 and Figure 1 we tabulate the function $k(X)$, defined by Pomerance, Selfridge and Wagstaff [6] by

$$C(X) = X \exp\left(-k(X) \frac{\log X \log\log\log X}{\log\log X}\right).$$

They proved that $\liminf k \geq 1$ and suggested that $\limsup k$ might be 2, although they also observed that within the range of their tables $k(X)$ is decreasing: Pomerance [7],[8] gave a heuristic argument suggesting that $\lim k = 1$. The decrease in k is reversed between 10^{13} and 10^{14}: see Figure 1. We find no clear support from our computations for any conjecture on a limiting value of k.

d	N
3	561
4	41041
5	825265
6	321197185
7	5394826801
8	232250619601
9	9746347772161
10	1436697831295441
11	60977817398996785
12	7156857700403137441
13	1791562810662585767521
14	87674969936234821377601
15	6553130926752006031481761
16	1590231231043178376951698401
17	35237869211718889547310642241
18	32809426840359564991177172754241
19	2810864562635368426005268142616001
20	349407515342287435050603204719587201
21	125861887849639969847638681038680787361
22	12758106140074522771498516740500829830401
23	2333379336546216408131111533710540349903201
24	294571791067375389885907239089503408618560001
25	130912961974316767723865201454187955056178415601
26	13513093081489380840188651246675032067011140079201
27	7482895937713262392883306949172917048928068129206401
28	1320340354477450170682291329830138947225695029536281601
29	379382381447399527322618466130154668512652910714224209601
30	70416887142533176417390411931483993124120785701395296424001
31	2884167509593581480205474627684686008624483147814647841436801
32	475486837760104673211993383998136308197201494852251082641778401
33	13347338771470623824869348071051978994960022011138499204965105416001
34	2608493230753718356697840943838121203592607838101572257306233883824001

TABLE 3. The smallest Carmichael numbers with d prime factors for d up to 34.

In Table 5 we also give the ratios $C(10^n)/C\big(10^{n-1}\big)$ investigated by Swift [10]. Swift's ratio, again initially decreasing, also increases again before 10^{15}.

In Table 6 and Figure 2 we see that within the range of our computations $C(X)$ is a slowly growing power of X: about $X^{0.339}$ for X around 10^{17}.

In Table 7 we give the number of Carmichael numbers in each class modulo m for $m = 5$, 7, 11 and 12.

In Tables 8 and 9 we give the number of Carmichael numbers divisible by primes p up to 97. In Table 8 we count all Carmichael numbers divisible by p: in Table 9 we count only those for which p is the smallest prime factor.

The largest prime factor of a Carmichael number up to 10^{17} is $p = 223401361$, dividing $n = 99816335969903281 = 17 \cdot 31 \cdot 379 \cdot 2237 \cdot p$. We have $p = n^{0.4911}$: indeed $n = p(2p-1)$ and $n - 1 = (p-1)(2p+1)$.

The largest prime to occur as the smallest prime factor of a Carmichael number in this range is 380251, dividing

$$90256390764228001 = 380251 \cdot 410761 \cdot 577981.$$

d	$\frac{\log(S_d)}{2\log 2(d-1)\log(d-1)}$
3	3.293621187
4	2.324869052
5	1.772215418
6	1.755823184
7	1.503593258
8	1.385943139
9	1.296862582
10	1.273113126
11	1.210794211
12	1.187291825
13	1.183842403
14	1.142841324
15	1.115637251
16	1.112255293
17	1.068846695
18	1.086833722
19	1.067861876
20	1.055266652
21	1.056211783
22	1.041906608
23	1.034833298
24	1.024202006
25	1.026042173
26	1.014073506
27	1.017122489
28	1.010169037
29	1.007224804
30	1.000945160
31	0.984182030
32	0.993535534
33	0.990337138
34	0.984854273

TABLE 4. The smallest Carmichael number with d prime factors S_d compared to $2\log 2(d-1)\log(d-1)$.

We note that this number is of the form $(25k+1)(27k+1)(38k+1)$ with $k = 15210$.

We define the *index* of a Carmichael number N to be the integer $i(N) = (N-1)/\lambda(N)$ where λ is the Carmichael function, the exponent of the multiplicative group. A result of Somer [9] implies that $i(N) \to \infty$ as N runs through Carmichael numbers. In Table 10 we list the Carmichael numbers known to have index less than 100: the list is complete for numbers up to 10^{17}.

We define the *Lehmer index* of a number N to be $\ell(N) = (N-1)/\phi(N)$. Lehmer's totient problem asks whether there is a number of integral Lehmer index greater than 1 (that is, whether there is a composite N such that $\phi(N)$ divides $N-1$). Such a number would have to be a Carmichael number. See Guy [2] §B37 for further details.

The Carmichael numbers up to 10^{17} with Lehmer index ≥ 2 are listed in Table 11.

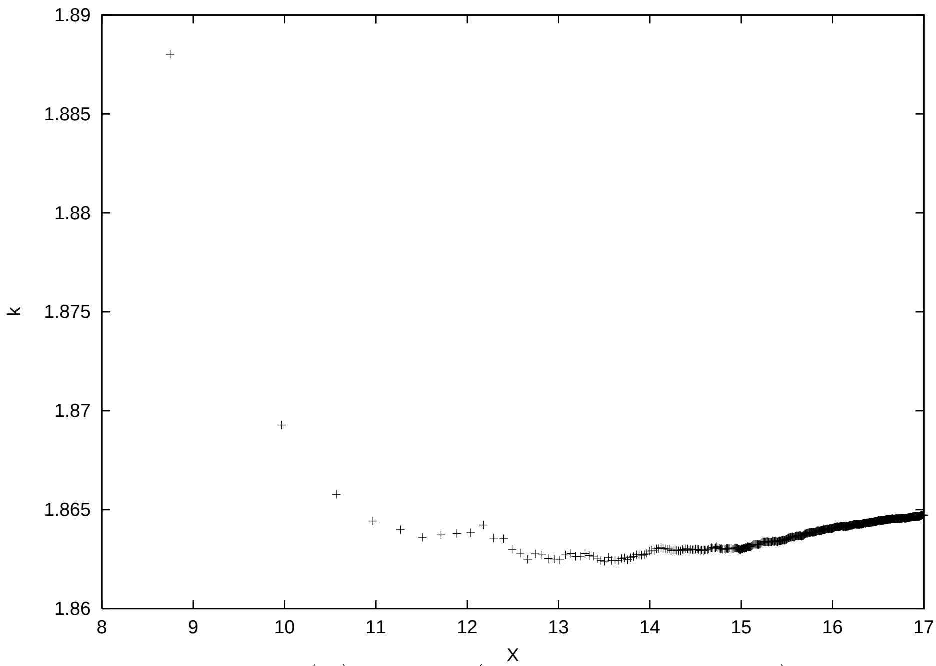

FIGURE 1. $k(X)$ versus X (expressed in powers of 10).

n	$k\,(10^n)$	$C(10^n)/C\big(10^{n-1}\big)$
3	2.93319	
4	2.19547	7.000
5	2.07632	2.286
6	1.97946	2.688
7	1.93388	2.441
8	1.90495	2.429
9	1.87989	2.533
10	1.86870	2.396
11	1.86421	2.330
12	1.86377	2.286
13	1.86240	2.339
14	1.86293	2.319
15	1.86301	2.353
16	1.86406	2.335
17	1.86472	2.373

TABLE 5. The function $k(X)$ and growth of $C(X)$ for $X = 10^n$, $n \leq 17$.

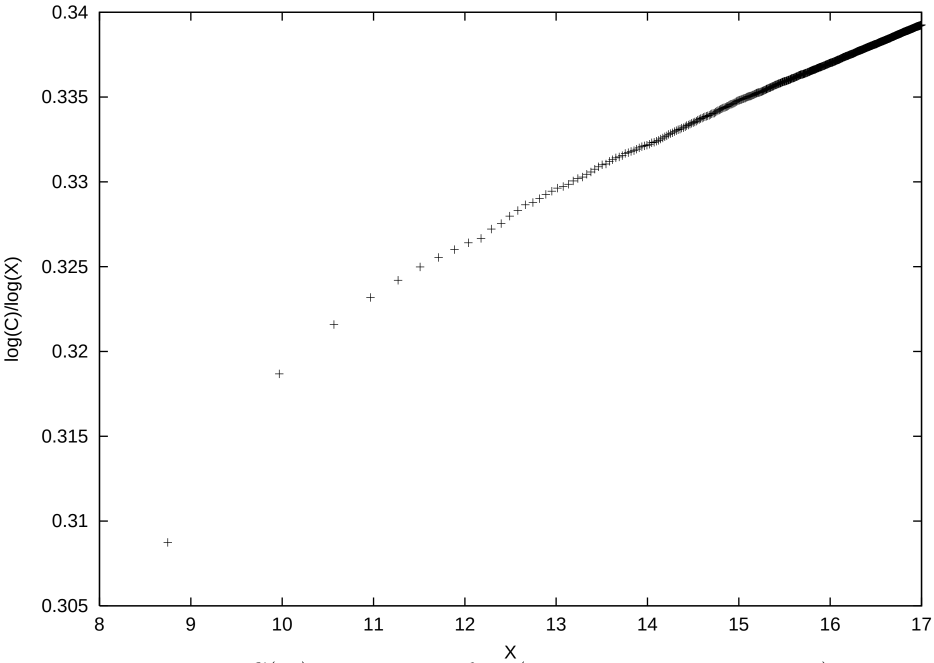

FIGURE 2. $C(X)$ as a power of X (expressed in powers of 10).

n	$\log C\left(10^n\right)/(n \log 10)$
4	.21127
5	.24082
6	.27224
7	.28874
8	.30082
9	.31225
10	.31895
11	.32336
12	.32633
13	.32962
14	.33217
15	.33480
16	.33700
17	.33926

TABLE 6. $C(X)$ as a power of X.

m	c	25.10^9	10^{11}	10^{12}	10^{13}	10^{14}	10^{15}	10^{16}	10^{17}
5	0	203	312	627	1330	2773	5814	12200	25523
	1	1652	2785	6575	15755	37467	90167	215713	520990
	2	82	154	327	702	1484	3048	6094	12912
	3	102	172	344	725	1463	3059	6285	12845
	4	124	182	368	767	1519	3124	6391	13085
7	0	401	634	1334	2774	5891	12691	27550	60602
	1	1096	1885	4613	11447	28001	69131	168856	414999
	2	105	186	432	967	2109	4599	10011	21961
	3	152	232	496	1055	2178	4707	10039	21918
	4	129	211	450	985	2122	4592	9944	21832
	5	138	222	454	1033	2224	4777	10125	21911
	6	142	235	462	1018	2181	4715	10158	22132
11	0	335	547	1324	3006	7032	16563	38576	90731
	1	640	1131	2770	6786	16548	40891	100071	248840
	2	139	217	473	1068	2361	5338	12054	27268
	3	142	220	457	1045	2348	5319	12186	27483
	4	104	187	442	1026	2317	5261	11917	27276
	5	152	243	466	1066	2370	5316	12194	27670
	6	116	198	440	1061	2400	5384	12155	27362
	7	122	195	458	1023	2223	5165	11853	26960
	8	129	222	475	1107	2450	5449	12012	27639
	9	131	218	465	1042	2285	5179	11835	27158
	10	153	227	471	1049	2372	5347	11830	26968
12	1	2071	3462	7969	18761	43760	103428	243382	579215
	3	0	0	1	2	2	5	5	7
	5	20	32	64	124	228	448	805	1470
	7	47	75	147	289	547	1027	1906	3589
	9	25	36	60	103	165	294	560	1018
	11	0	0	0	0	4	10	25	56

TABLE 7. The number of Carmichael numbers in each class modulo m for $m = 5$, 7, 11 and 12.

p	25.10^9	10^{11}	10^{12}	10^{13}	10^{14}	10^{15}	10^{16}	10^{17}
3	25	36	61	105	167	299	565	1025
5	203	312	627	1330	2773	5814	12200	25523
7	401	634	1334	2774	5891	12691	27550	60602
11	335	547	1324	3006	7032	16563	38576	90731
13	483	807	1784	3998	9045	20758	47785	110320
17	293	489	1182	2817	6640	16019	38302	91783
19	372	608	1355	3345	7797	18638	44389	106273
23	113	207	507	1282	3135	7716	18867	46612
29	194	336	832	2094	5158	12721	31110	76647
31	335	571	1320	3086	7270	17382	41440	99845
37	320	535	1270	2926	6826	16220	38647	92744
41	227	390	1001	2418	5896	14344	34759	84977
43	184	296	772	1920	4663	11594	28650	70904
47	53	80	199	492	1223	2873	6810	17002
53	92	160	351	813	2041	5143	12256	30560
59	26	41	92	262	644	1611	3959	9613
61	269	453	1075	2542	6047	14429	34503	83268
67	110	178	407	1063	2540	6306	15295	37521
71	104	194	521	1320	3351	8546	21485	53857
73	198	348	849	2145	4925	11929	29072	71137
79	64	107	247	686	1728	4318	10693	26766
83	14	24	56	137	340	838	1929	4640
89	68	131	320	788	1951	4981	12178	30737
97	123	193	495	1277	3123	7594	18706	45542

TABLE 8. Primes occurring in Carmichael numbers.

p	25.10^9	10^{11}	10^{12}	10^{13}	10^{14}	10^{15}	10^{16}	10^{17}
3	25	36	61	105	167	299	565	1025
5	202	309	624	1325	2765	5797	12175	25481
7	364	579	1218	2557	5461	11874	25915	57459
11	263	428	1071	2509	5979	14397	33893	80745
13	237	431	1058	2462	5699	13514	32025	76256
17	117	206	496	1318	3244	8114	20206	50170
19	152	244	532	1401	3358	8141	20020	49413
23	37	78	207	535	1360	3317	8195	20803
29	55	103	284	729	1822	4659	11577	29149
31	101	168	390	876	2116	5153	12575	30667
37	60	95	219	551	1401	3418	8594	21382
41	35	68	171	414	1092	2736	6788	17275
43	35	65	168	403	943	2308	5520	13636
47	14	16	36	81	195	459	1135	2854
53	19	30	55	147	363	973	2327	5842
59	2	4	11	43	100	272	618	1542
61	34	58	148	364	851	1978	4722	11278
67	8	18	50	123	317	815	1950	4843
71	15	25	66	161	389	979	2480	6178
73	14	28	68	175	406	1015	2508	6277
79	4	10	17	66	175	467	1163	2873
83	1	1	4	8	39	79	175	457
89	10	16	23	55	148	409	1003	2523
97	10	20	50	106	261	606	1413	3445

TABLE 9. Primes occurring as least prime factor in Carmichael numbers.

i	N	factors
5	6601	$7 \cdot 23 \cdot 41$
7	561	$3 \cdot 11 \cdot 17$
18	42018333841	$11 \cdot 47 \cdot 1049 \cdot 77477$
18	55462177	$17 \cdot 23 \cdot 83 \cdot 1709$
18	8885251441	$11 \cdot 47 \cdot 1109 \cdot 15497$
21	10585	$5 \cdot 29 \cdot 73$
22	2465	$5 \cdot 17 \cdot 29$
23	1105	$5 \cdot 13 \cdot 17$
25	11921001	$3 \cdot 29 \cdot 263 \cdot 521$
31	62745	$3 \cdot 5 \cdot 47 \cdot 89$
37	11972017	$43 \cdot 433 \cdot 643$
37	67902031	$43 \cdot 271 \cdot 5827$
39	334153	$19 \cdot 43 \cdot 409$
43	52633	$7 \cdot 73 \cdot 103$
44	15841	$7 \cdot 31 \cdot 73$
45	8911	$7 \cdot 19 \cdot 67$
47	2821	$7 \cdot 13 \cdot 31$
48	1729	$7 \cdot 13 \cdot 19$
49	1208361237478669	$53 \cdot 653 \cdot 26479 \cdot 1318579$
50	4199932801	$29 \cdot 499 \cdot 503 \cdot 577$
52	206955841	$17 \cdot 71 \cdot 277 \cdot 619$
53	1271325841	$17 \cdot 31 \cdot 179 \cdot 13477$
54	4169867689	$13 \cdot 29 \cdot 383 \cdot 28879$
55	271794601	$13 \cdot 19 \cdot 743 \cdot 1481$
60	6840001	$7 \cdot 17 \cdot 229 \cdot 251$
61	1962804565	$5 \cdot 103 \cdot 149 \cdot 25579$
67	11985924995083901	$29 \cdot 101 \cdot 1427 \cdot 16349 \cdot 175403$
67	410041	$41 \cdot 73 \cdot 137$
70	162401	$17 \cdot 41 \cdot 233$
76	4752717761	$11 \cdot 17 \cdot 107 \cdot 173 \cdot 1373$
81	35575075809505	$5 \cdot 197 \cdot 223 \cdot 353 \cdot 458807$
82	1496405933740345	$5 \cdot 47 \cdot 317 \cdot 40253 \cdot 499027$
83	142159958924185	$5 \cdot 37 \cdot 107 \cdot 58379 \cdot 123017$
83	158664761899885	$5 \cdot 37 \cdot 107 \cdot 53987 \cdot 148469$
83	204370370140285	$5 \cdot 37 \cdot 107 \cdot 48677 \cdot 212099$
83	24831908105124205	$5 \cdot 29 \cdot 719 \cdot 3023 \cdot 78790717$
85	171189355538562901	$23 \cdot 71 \cdot 983 \cdot 1031 \cdot 103437589$
90	3778118040573702001	$11 \cdot 47 \cdot 1051 \cdot 67967 \cdot 102302009$
92	520178982961	$11 \cdot 29 \cdot 131 \cdot 607 \cdot 20507$
94	12782849065	$5 \cdot 7 \cdot 269 \cdot 317 \cdot 4283$
95	8956911601	$11 \cdot 17 \cdot 127 \cdot 131 \cdot 2879$
97	5472940991761	$199 \cdot 241 \cdot 863 \cdot 132233$
97	721574219707441	$167 \cdot 241 \cdot 5039 \cdot 3557977$
97	83565865434172201	$103 \cdot 1993 \cdot 9551 \cdot 42622169$
97	9729822470631481	$127 \cdot 409 \cdot 110681 \cdot 1692407$
99	438253965870337	$43 \cdot 139 \cdot 49409 \cdot 1484009$

TABLE 10. Carmichael numbers with small index

ℓ	N	factors
2.00163	3852971941960065	$3 \cdot 5 \cdot 23 \cdot 89 \cdot 113 \cdot 1409 \cdot 788129$
2.00388	655510549443465	$3 \cdot 5 \cdot 23 \cdot 53 \cdot 389 \cdot 2663 \cdot 34607$
2.00837	13462627333098945	$3 \cdot 5 \cdot 23 \cdot 53 \cdot 197 \cdot 8009 \cdot 466649$
2.01001	26708253318968145	$3 \cdot 5 \cdot 17 \cdot 113 \cdot 57839 \cdot 16025297$
2.03067	269040992399565	$3 \cdot 5 \cdot 23 \cdot 29 \cdot 4637 \cdot 5799149$
2.03207	158353658932305	$3 \cdot 5 \cdot 17 \cdot 89 \cdot 149 \cdot 563 \cdot 83177$
2.03614	1817671359979245	$3 \cdot 5 \cdot 23 \cdot 29 \cdot 359 \cdot 11027 \cdot 45893$
2.07138	16057190782234785	$3 \cdot 5 \cdot 17 \cdot 29 \cdot 269 \cdot 6089 \cdot 1325663$
2.07294	75131642415974145	$3 \cdot 5 \cdot 23 \cdot 29 \cdot 53 \cdot 617 \cdot 9857 \cdot 23297$
2.08129	881715504450705	$3 \cdot 5 \cdot 17 \cdot 47 \cdot 89 \cdot 113 \cdot 503 \cdot 14543$
2.08381	31454143858820145	$3 \cdot 5 \cdot 17 \cdot 23 \cdot 2129 \cdot 39293 \cdot 64109$
2.08894	6128613921672705	$3 \cdot 5 \cdot 17 \cdot 23 \cdot 353 \cdot 7673 \cdot 385793$
2.09841	12301576752408945	$3 \cdot 5 \cdot 23 \cdot 29 \cdot 53 \cdot 113 \cdot 197 \cdot 1042133$
2.11432	1886616373665	$3 \cdot 5 \cdot 17 \cdot 23 \cdot 83 \cdot 353 \cdot 10979$
2.11493	3193231538989185	$3 \cdot 5 \cdot 17 \cdot 23 \cdot 113 \cdot 167 \cdot 2927 \cdot 9857$
2.13936	11947816523586945	$3 \cdot 5 \cdot 17 \cdot 23 \cdot 89 \cdot 113 \cdot 233 \cdot 617 \cdot 1409$

TABLE 11. Carmichael numbers with Lehmer index greater than 2

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2 ELDON ROAD, CHELTENHAM, GLOS GL52 6TU, U.K.

E-mail address: `rgep@chalcedon.demon.co.uk`