THE CARMICHAEL NUMBERS UP TO 10^{16}

RICHARD G.E. PINCH

Abstract. We extend our previous computations to show that there are 246683 Carmichael numbers up to 10^{16}. As before, the numbers were generated by a back-tracking search for possible prime factorisations together with a "large prime variation". We present further statistics on the distribution of Carmichael numbers.

1. Introduction

A *Carmichael number* N is a composite number N with the property that for every b prime to N we have $b^{N-1} \equiv 1 \mod N$. It follows that a Carmichael number N must be square-free, with at least three prime factors, and that $p-1|N-1$ for every prime p dividing N: conversely, any such N must be a Carmichael number.

For background on Carmichael numbers and details of previous computations we refer to our previous paper [2]: in that paper we described the computation of the Carmichael numbers up to 10^{15} and presented some statistics. These computations have since been extended to 10^{16}, using the same techniques, and we present further statistics.

The complete list of Carmichael numbers up to 10^{16} is available by anonymous FTP from `ftp.dpmms.cam.ac.uk` in directory `/pub/Carmichael`.

2. Statistics

We have shown that there are 246683 Carmichael numbers up to 10^{16}, all with at most 10 prime factors. We let $C(X)$ denote the number of Carmichael numbers less than X and $C(d, X)$ denote the number with exactly d prime factors. Table 1 gives the values of $C(X)$ and Table 2 the values of $C(d, X)$ for X in powers of 10 up to 10^{16}.

We have used the same methods to calculate the smallest Carmichael numbers with d prime factors for d up to 20. The results are given in Table 3.

In Table 4 and Figure 1 we tabulate the function $k(X)$, defined by Pomerance, Selfridge and Wagstaff [5] by

$$C(X) = X \exp\left(-k(X)\frac{\log X \log\log\log X}{\log\log X}\right).$$

They proved that $\liminf k \geq 1$ and suggested that $\limsup k$ might be 2, although they also observed that within the range of their tables $k(X)$ is decreasing: Pomerance [3],[4] gave a heuristic argument suggesting that $\lim k = 1$. The decrease in k is reversed between 10^{13} and 10^{14}: see Figure 1. We find no clear support from our computations for any conjecture on a limiting value of k.

In Table 4 we also give the ratios $C(10^n)/C(10^{n-1})$ investigated by Swift [6]. Swift's ratio, again initially decreasing, also increases again before 10^{15}.

In Table 5 and Figure 2 we see that within the range of our computations $C(X)$ is a slowly growing power of X: about $X^{0.337}$ for X around 10^{16}.

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n	$C(10^n)$
3	1
4	7
5	16
6	43
7	105
8	255
9	646
10	1547
11	3605
12	8241
13	19279
14	44706
15	105212
16	246683

TABLE 1. Distribution of Carmichael numbers up to 10^{16}.

X	3	4	5	6	7	8	9	10	total
3	1	0	0	0	0	0	0	0	1
4	7	0	0	0	0	0	0	0	7
5	12	4	0	0	0	0	0	0	16
6	23	19	1	0	0	0	0	0	43
7	47	55	3	0	0	0	0	0	105
8	84	144	27	0	0	0	0	0	255
9	172	314	146	14	0	0	0	0	646
10	335	619	492	99	2	0	0	0	1547
11	590	1179	1336	459	41	0	0	0	3605
12	1000	2102	3156	1714	262	7	0	0	8241
13	1858	3639	7082	5270	1340	89	1	0	19279
14	3284	6042	14938	14401	5359	655	27	0	44706
15	6083	9938	29282	36907	19210	3622	170	0	105212
16	10816	16202	55012	86696	60150	16348	1436	23	246683

TABLE 2. Values of $C(X)$ and $C(d,X)$ for $d \leq 10$ and X in powers of 10 up to 10^{16}.

In Table 6 we give the number of Carmichael numbers in each class modulo m for $m = 5$, 7, 11 and 12.

In Tables 7 and 8 we give the number of Carmichael numbers divisible by primes p up to 97. In Table 7 we count all Carmichael numbers divisible by p: in Table 8 we count only those for which p is the smallest prime factor.

The largest prime factor of a Carmichael number up to 10^{16} is 68786257, dividing

$$9463098235353841 = 13 \cdot 31 \cdot 541 \cdot 631 \cdot 68786257$$

and the largest prime to occur as the smallest prime factor of a Carmichael number in this range is 174763, dividing

$$9585921133193329 = 174763 \cdot 199729 \cdot 274627.$$

We note that this number is of the form $(7k+1)(8k+1)(11k+1)$ with $k = 24966$.

d	N factors
3	561 3.11.17
4	41041 7.11.13.41
5	825265 5. 7.17.19.73
6	321197185 5.19.23.29.37.137
7	5394826801 7.13.17.23.31.67.73
8	232250619601 7.11.13.17.31.37.73.163
9	9746347772161 7.11.13.17.19.31.37.41.641
10	1436697831295441 11.13.19.29.31.37.41.43.71.127
11	60977817398996785 5. 7.17.19.23.37.53.73.79.89.233
12	7156857700403137441 11.13.17.19.29.37.41.43.61.97.109.127
13	1791562810662585767521 11.13.17.19.31.37.43.71.73.97.109.113.127
14	87674969936234821377601 7.13.17.19.23.31.37.41.61.67.89.163.193.241
15	6553130926752006031481761 11.13.17.19.29.31.41.43.61.71.73.109.113.127.181
16	1590231231043178376951698401 17.19.23.29.31.37.41.43.61.67.71.73.79. 97.113.199
17	35237869211718889547310642241 13.17.19.23.29.31.37.41.43.61.67.71.73. 97.113.127.211
18	32809426840359564991177172754241 13.17.19.23.29.31.37.41.43.61.67.71.73. 97.127.199.281.397
19	2810864562635368426005268142616001 13.17.19.23.29.31.37.41.43.61.67.71.73.109.113.127.151.281.353
20	349407515342287435050603204719587201 11.13.17.19.29.31.37.41.43.61.71.73.97.101.109.113.151.181.193.641

TABLE 3. The smallest Carmichael numbers with d prime factors for d up to 20.

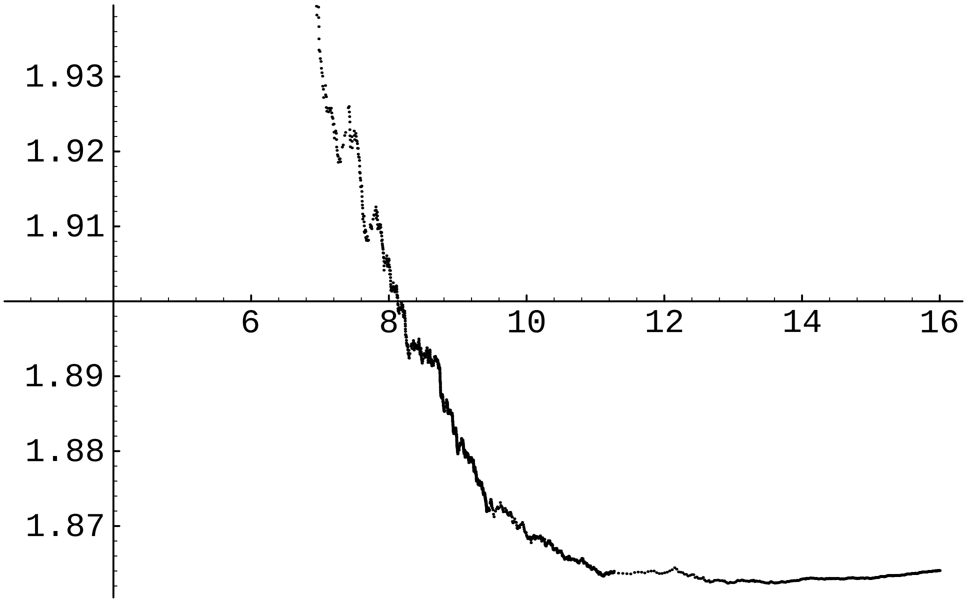

FIGURE 1. $k(X)$ versus X (expressed in powers of 10).

n	$k(10^n)$	$C(10^n)/C(10^{n-1})$
3	2.93319	
4	2.19547	7.000
5	2.07632	2.286
6	1.97946	2.688
7	1.93388	2.441
8	1.90495	2.429
9	1.87989	2.533
10	1.86870	2.396
11	1.86421	2.330
12	1.86377	2.286
13	1.86240	2.339
14	1.86293	2.319
15	1.86301	2.353
16	1.86406	2.335

TABLE 4. The function $k(X)$ and growth of $C(X)$ for $X = 10^n$, $n \leq 16$.

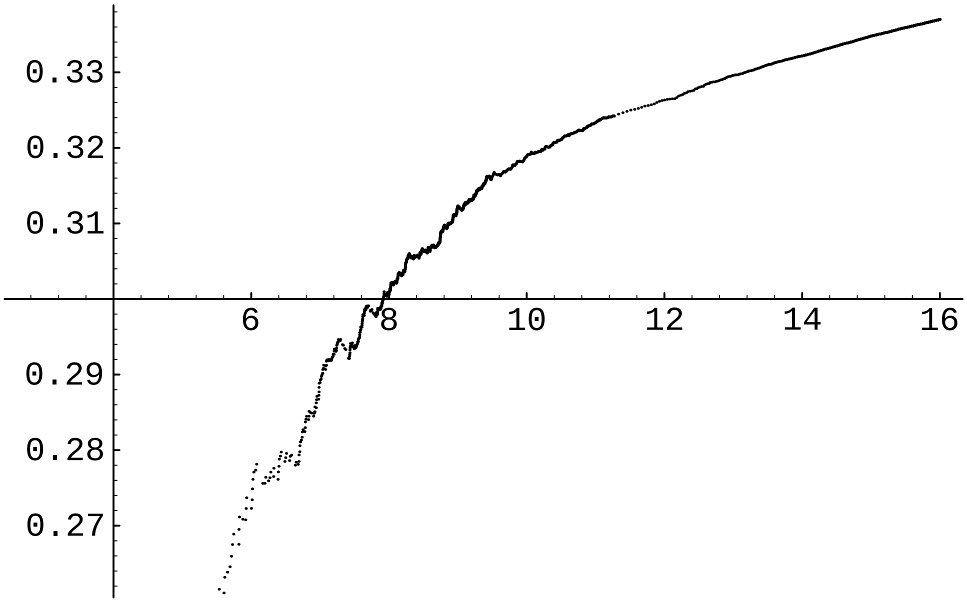

FIGURE 2. $C(X)$ as a power of X (expressed in powers of 10).

n	$\log C\left(10^n\right)/(n \log 10)$
4	.21127
5	.24082
6	.27224
7	.28874
8	.30082
9	.31225
10	.31895
11	.32336
12	.32633
13	.32962
14	.33217
15	.33480
16	.33700

TABLE 5. $C(X)$ as a power of X.

m	c	25.10^9	10^{11}	10^{12}	10^{13}	10^{14}	10^{15}	10^{16}
5	0	203	312	627	1330	2773	5814	12200
	1	1652	2785	6575	15755	37467	90167	215713
	2	82	154	327	702	1484	3048	6094
	3	102	172	344	725	1463	3059	6285
	4	124	182	368	767	1519	3124	6391
7	0	401	634	1334	2774	5891	12691	27550
	1	1096	1885	4613	11447	28001	69131	168856
	2	105	186	432	967	2109	4599	10011
	3	152	232	496	1055	2178	4707	10039
	4	129	211	450	985	2122	4592	9944
	5	138	222	454	1033	2224	4777	10125
	6	142	235	462	1018	2181	4715	10158
11	0	335	547	1324	3006	7032	16563	38576
	1	640	1131	2770	6786	16548	40891	100071
	2	139	217	473	1068	2361	5338	12054
	3	142	220	457	1045	2348	5319	12186
	4	104	187	442	1026	2317	5261	11917
	5	152	243	466	1066	2370	5316	12194
	6	116	198	440	1061	2400	5384	12155
	7	122	195	458	1023	2223	5165	11853
	8	129	222	475	1107	2450	5449	12012
	9	131	218	465	1042	2285	5179	11835
	10	153	227	471	1049	2372	5347	11830
12	1	2071	3462	7969	18761	43760	103428	243382
	3	0	0	1	2	2	5	5
	5	20	32	64	124	228	448	805
	7	47	75	147	289	547	1027	1906
	9	25	36	60	103	165	294	560
	11	0	0	0	0	4	10	25

TABLE 6. The number of Carmichael numbers in each class modulo m for $m = 5$, 7, 11 and 12.

p	25.10^9	10^{11}	10^{12}	10^{13}	10^{14}	10^{15}	10^{16}
3	25	36	61	105	167	299	565
5	203	312	627	1330	2773	5814	12200
7	401	634	1334	2774	5891	12691	27550
11	335	547	1324	3006	7032	16563	38576
13	483	807	1784	3998	9045	20758	47785
17	293	489	1182	2817	6640	16019	38302
19	372	608	1355	3345	7797	18638	44389
23	113	207	507	1282	3135	7716	18867
29	194	336	832	2094	5158	12721	31110
31	335	571	1320	3086	7270	17382	41440
37	320	535	1270	2926	6826	16220	38647
41	227	390	1001	2418	5896	14344	34759
43	184	296	772	1920	4663	11594	28650
47	53	80	199	492	1223	2873	6810
53	92	160	351	813	2041	5143	12256
59	26	41	92	262	644	1611	3959
61	269	453	1075	2542	6047	14429	34503
67	110	178	407	1063	2540	6306	15295
71	104	194	521	1320	3351	8546	21485
73	198	348	849	2145	4925	11929	29072
79	64	107	247	686	1728	4318	10693
83	14	24	56	137	340	838	1929
89	68	131	320	788	1951	4981	12178
97	123	193	495	1277	3123	7594	18706

TABLE 7. Primes occurring in Carmichael numbers.

p	25.10^9	10^{11}	10^{12}	10^{13}	10^{14}	10^{15}	10^{16}
3	25	36	61	105	167	299	565
5	202	309	624	1325	2765	5797	12175
7	364	579	1218	2557	5461	11874	25915
11	263	428	1071	2509	5979	14397	33893
13	237	431	1058	2462	5699	13514	32025
17	117	206	496	1318	3244	8114	20206
19	152	244	532	1401	3358	8141	20020
23	37	78	207	535	1360	3317	8195
29	55	103	284	729	1822	4659	11577
31	101	168	390	876	2116	5153	12575
37	60	95	219	551	1401	3418	8594
41	35	68	171	414	1092	2736	6788
43	35	65	168	403	943	2308	5520
47	14	16	36	81	195	459	1135
53	19	30	55	147	363	973	2327
59	2	4	11	43	100	272	618
61	34	58	148	364	851	1978	4722
67	8	18	50	123	317	815	1950
71	15	25	66	161	389	979	2480
73	14	28	68	175	406	1015	2508
79	4	10	17	66	175	467	1163
83	1	1	4	8	39	79	175
89	10	16	23	55	148	409	1003
97	10	20	50	106	261	606	1413

TABLE 8. Primes occurring as least prime factor in Carmichael numbers.

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Queens' College, Silver Street, Cambridge CB3 9ET, U.K.
E-mail address: `rgep@cam.ac.uk`